

UNIFORM SYMBOLIC TOPOLOGIES VIA MULTINOMIAL EXPANSIONS

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ABSTRACT. Over an arbitrary field \mathbb{F} , Harbourne (Conj. 8.4.2 of [3]) conjectured that the symbolic power $I^{(N(r-1)+1)} \subseteq I^r$ for all $r > 0$ and all graded ideals I in $S = \mathbb{F}[\mathbb{P}^N] = \mathbb{F}[x_0, \dots, x_N]$ ($N \geq 2$). The conjecture has been disproven in both zero- and odd prime characteristic ([5, 8], see also [1]). However, the conjecture does hold over any field when, e.g., I is a monomial ideal in S .

This manuscript is a sequel to the proof of Theorems 1.1 and 3.1 in [16]. Working with a prescribed type of prime ideal Q inside of, e.g., tensor products of domains which are of finite type over an algebraically closed field \mathbb{F} , we present binomial- and multinomial expansion criteria for containments of type $Q^{(Er)} \subseteq Q^r$, or even better, of type $Q^{(E(r-1)+1)} \subseteq Q^r$ for all $r > 0$. In the closing section, we indicate how to specify explicit multipliers E , when such are known.

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1. INTRODUCTION AND CONVENTIONS FOR THE PAPER

Over an arbitrary field \mathbb{F} , $S = \mathbb{F}[\mathbb{P}^N] = \mathbb{F}[x_0, x_1, \dots, x_N]$ is a standard N -graded polynomial ring. The groundbreaking work of Ein-Lazarsfeld-Smith and Hochster-Huneke [6, 9] implies that the symbolic power $I^{(Nr)} \subseteq I^r$ for all graded ideals $0 \subsetneq I \subsetneq S$ and all integers $r > 0$. In particular, $I^{(4)} \subseteq I^2$ holds for all graded ideals in $\mathbb{F}[\mathbb{P}^2]$, and Huneke asked whether an improvement $I^{(3)} \subseteq I^2$ holds for any radical ideal I defining a finite set of points in \mathbb{P}^2 . Building on this, Harbourne ([3, Conj. 8.4.2]) proposed dropping the symbolic power from Nr down to the **Harbourne-Huneke bound** $Nr - (N - 1) = N(r - 1) + 1$ when $N \geq 2$, i.e.,

$$I^{(N(r-1)+1)} \subseteq I^r \text{ for any graded ideal } 0 \subsetneq I \subsetneq S, \text{ all } r > 0, \text{ and all } N \geq 2. \quad (1.0.1)$$

However, Dumnicki, Szemberg, and Tutaj-Gasińska showed in characteristic zero [5] that the containment $I^{(3)} \subseteq I^2$ can fail for a radical ideal defining a point configuration in \mathbb{P}^2 . Harbourne-Seceleanu showed in odd positive characteristic [8] that (1.0.1) can fail for pairs $(N, r) \neq (2, 2)$ and ideals I defining a point configuration in \mathbb{P}^N . Akeseh [1] cooks up many new counterexamples to (1.0.1) from these original constructions. No prime ideal counterexample has been found.

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Our goal is to establish Harbourne-Huneke bounds on the growth of symbolic powers of certain primes in non-regular Noetherian rings containing a field. This project first began with Theorems 1.1 and 3.1 in [16]: to clarify, normal affine semigroup rings are domains generated by Laurent monomials that arise as the coordinate rings of normal affine toric varieties.

Theorem 1.1 (Thm. 1.1 in [16]). *Let R_1, \dots, R_n be normal affine semigroup rings over a field \mathbb{F} , built, respectively, from full-dimensional strongly convex rational polyhedral cones $\sigma_i \subseteq \mathbb{R}^{m_i}$, $1 \leq i \leq n$. For each $1 \leq i \leq n$, suppose there is an integer $D_i > 0$ such that $P^{(D_i(r-1)+1)} \subseteq P^r$ for all $r > 0$ and all monomial primes $P \subseteq R_i$. Set $D := \max\{D_1, \dots, D_n\}$. Then $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$ and any monomial prime Q in the normal affine semigroup ring $R = R_1 \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} R_n$.*

In proving (1.1), we needed to know first that monomial primes in R_i expand to monomial primes in R , that any Q as above can be expressed as a sum $Q = \sum_{i=1}^n P_i R$ where each $P_i \subseteq R_i$ is a monomial prime, and that the symbolic powers of Q admit a multinomial expansion in terms of symbolic powers of the $P_i R$. These ideas will resurge below, but in a more general setup.

One drawback of (1.1) is that it only covers a finite collection of prime ideals. What follows is the main result of this paper, a more powerful variant of (1.1) that can cover infinitely-many primes inside of tensor product domains, as long as some tensor factor has dimension at least two:

Theorem 1.2. *Let \mathbb{F} be an algebraically closed field. Let R_1, \dots, R_n ($n \geq 2$) be affine commutative \mathbb{F} -algebras which are domains. Suppose that for each $1 \leq i \leq n$, there exists a positive integer D_i such that for all prime ideals P in R_i , either: (1) $P^{(D_i r)} \subseteq P^r$ for all $r > 0$; or, even stronger, (2) $P^{(D_i(r-1)+1)} \subseteq P^r$ for all $r > 0$. Fix any n prime ideals P_i in R_i , and consider the expanded ideals $P'_i = P_i T$ in the affine domain $T = (\bigotimes_{\mathbb{F}} R_i)_{i=1}^n$, along with their sum $Q = \sum_{i=1}^n P'_i$ in T . When (1) holds for all i , $Q^{(Dr)} \subseteq Q^r$ for all $r > 0$, where $D = D_1 + \dots + D_n$. When (2) holds for all i , this improves to $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$, where $D = \max\{D_1, \dots, D_n\}$.*

The proof of this theorem leverages a multinomial formula for the symbolic powers of the prime ideal Q in T (Theorem (2.8)). Hà, Nguyen, Trung, and Trung recently announced a binomial theorem for symbolic powers of ideal sums [7, Thm. 3.4], generalizing [4, Thm. 7.8], where one takes two arbitrary ideals $I \subseteq A, J \subseteq B$ inside of two Noetherian commutative algebras over a common field k , whose tensor product $R = A \otimes_k B$ is Noetherian; see Remark (2.10). However, we give a proof of the multinomial theorem (2.8) which is more elementary and self-contained.

Conventions: All our rings are Noetherian and commutative with identity. Indeed, our rings will typically be *affine* \mathbb{F} -algebras, that is, of finite type over a fixed field \mathbb{F} of arbitrary characteristic. By *algebraic variety*, we will mean an integral scheme of finite type over the field \mathbb{F} .

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2. A MULTINOMIAL THEOREM FOR SYMBOLIC POWERS OF PRIMES

If P is any prime ideal in a Noetherian ring R , its a -**th** ($a \in \mathbb{Z}_{>0}$) **symbolic power ideal**

$$P^{(a)} = P^a R_P \cap R = \{f \in R : uf \in P^a \text{ for some } u \in R - P\}$$

is the P -primary component in any Lasker-Noether minimal primary decomposition of P^a ; it is the smallest P -primary ideal containing P^a . We separately set $P^{(0)} = P^0 = R$ to be the unit ideal. Note that $P^{(1)} = P$, while the inclusion $P^{(a)} \supseteq P^a$ for each $a > 1$ can be strict. Before proceeding, we record a handy asymptotic conversion lemma used to prove Theorem (1.1) in [16]; together with Proposition (2.6), it will be used to prove Theorem (1.2) later.

Lemma 2.1 (Cf., Lem. 3.3 in [16]). *Given any prime ideal P in a Noetherian ring $S \neq 0$, and $E \in \mathbb{Z}_{\geq 0}$,*

$$P^{(N)} \subseteq P^{[N/E]} \text{ for all } N \geq 0 \iff P^{(E(r-1)+1)} \subseteq P^r \text{ for all } r > 0.$$

Torsion free modules over Noetherian Domains. A module M over a domain R is **torsion free** if whenever $rx = 0$ for some $x \in M$ and $r \in R$, then either $r = 0$ or $x = 0$. We first record a lemma on torsion free modules to be used both here and in the next subsection (cf., Lemmas 15.6.7-8 from the Stacks Project page [15] on torsion free modules):

Lemma 2.2. *Let R be a Noetherian domain. Let M be a nonzero finitely generated R -module. Then the following assertions are equivalent:*

- (1) M is torsion free;
- (2) M is a submodule of a finitely generated free module;
- (3) (0) is the only associated prime of M , i.e., $\text{Ass}_R(M) = \{(0)\}$.

Working over an arbitrary field \mathbb{F} , we fix two affine \mathbb{F} -algebras R and S which are domains. The tensor product $T = R \otimes_{\mathbb{F}} S$ will be an affine \mathbb{F} -algebra. T is a domain when \mathbb{F} is algebraically closed (Milne [13, Prop. 4.15]). We note that when R and S are dully nice (e.g., polynomial, or normal toric rings more generally), T is a domain over any field. Our immediate goal here is to record a consequence of Lemma (2.3) below that will be important in the next subsection.

Lemma 2.3. *Suppose that all three of R , S , and $T = R \otimes_{\mathbb{F}} S$ are affine domains over a field \mathbb{F} . If M and N are finitely generated torsion free modules over R and S , respectively, then $M \otimes_{\mathbb{F}} N$ is a finitely generated torsion free T -module.*

Proof. Viewed as vector spaces, $M \otimes_{\mathbb{F}} N = 0$ if and only if $M = 0$ or $N = 0$, in which case torsion freeness is vacuous. So we will assume all three of M, N , and $M \otimes_{\mathbb{F}} N$ are nonzero. Per Lemma (2.2), suppose we have embeddings $M \subseteq R^a$ and $N \subseteq S^b$. Apply the functor $\bullet \otimes_{\mathbb{F}} N$ to the first inclusion to get $M \otimes N \subseteq R^a \otimes N$, which in turn is contained in $R^a \otimes S^b$ by tensoring the inclusion $N \subseteq S^b$ with R^a . Thus $M \otimes N \subseteq R^a \otimes S^b \cong (R \otimes S)^{ab} = T^{ab}$, where the isomorphism is easily checked in the category of \mathbb{F} -vector spaces since direct sum commutes with tensor product. Of course, this inclusion holds in the category of T -modules, and all T -submodules of T^{ab} are finitely generated since T is Noetherian, so we are done by invoking Lemma (2.2) again. \square

Lemma 2.4. *For any prime P in any Noetherian ring A , the finitely generated module $P^{(a)}/P^{(a+1)}$ is torsion free as an A/P -module for all integers $a \geq 0$.*

Proof. Say $\bar{x} \in (P^{(a)}/P^{(a+1)})$ is killed by $\bar{r} \in A/P$. This means, lifting to A , that $x \in P^{(a)}$ and $rx \in P^{(a+1)}$. Localize at P . Then $rx \in P^{(a+1)}A_P = P^{a+1}A_P$. If $r \notin P$, this means $x \in P^{a+1}A_P \cap A = P^{(a+1)}$. That is, either $\bar{r} = 0$ in A/P or otherwise, $\bar{x} = 0$ in $(P^{(a)}/P^{(a+1)})$. Ergo by definition, $(P^{(a)}/P^{(a+1)})$ is a torsion-free A/P -module. \square

Proposition 2.5. *Suppose that all three of R , S , and $T = R \otimes_{\mathbb{F}} S$ are affine domains over a field \mathbb{F} . Fix two prime ideals P and Q in R and S respectively, such that the affine \mathbb{F} -algebra $T' = (R/P) \otimes_{\mathbb{F}} (S/Q)$ is a domain. Then $(P^{(a)}/P^{(a+1)}) \otimes_{\mathbb{F}} (Q^{(b)}/Q^{(b+1)})$ is finitely generated and torsion free over T' for any pair of nonnegative integers a and b .*

Proving the Multinomial Theorem. Working over an algebraically closed field \mathbb{F} , we fix two affine \mathbb{F} -algebras R and S that are domains, and two prime ideals $P \subseteq R$, $Q \subseteq S$. Let

$$T = R \otimes S \supseteq P \otimes S + R \otimes Q =: PT + QT, \quad T' = (R/P) \otimes (S/Q) \cong T/(PT + QT),$$

where all tensor products are over \mathbb{F} . Both T and T' are affine domains over \mathbb{F} . The extended ideals PT, QT are both prime, along with their sum $PT + QT$. In stating that each of $PT + QT$, PT , QT will be prime, we cannot relax our hypothesis on \mathbb{F} to being perfect. For instance, \mathbb{R} is perfect (being of characteristic zero), and along the ring extension

$$S := \frac{\mathbb{R}[x]}{(x^2 + 1)} \cong \mathbb{C} \hookrightarrow T := \frac{\mathbb{C}[x]}{(x^2 + 1)} \cong \mathbb{C} \otimes_{\mathbb{R}} S \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$$

the zero ideal of S (which is maximal) extends to a radical ideal which is not primary, i.e., not prime. Thus the sum $(0)T = (0)S + (0)S$ of the extended zero ideals is not primary.

We now prove a binomial theorem for the symbolic powers of $PT + QT$. But first, we state a proposition. Relative to a flat map $\phi: A \rightarrow B$ of Noetherian rings, we define the ideal $JB := \langle \phi(J) \rangle B$ for any ideal J in A , and $J^r B = (JB)^r$ for all $r \geq 0$ since the two ideals share a generating set. We define a set $\mathcal{P}(A) = \{\text{prime ideals } P \subseteq A: PB \text{ is prime}\}$ consisting of prime ideals that extend along ϕ to prime ideals of B . When B is a polynomial ring in finitely many variables over A and ϕ is inclusion, $\mathcal{P}(A) = \text{Spec}(A)$. However, our example $T \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ shows that extensions of prime ideals may fail to be prime along an arbitrary faithfully flat extension of Noetherian rings.

Proposition 2.6 (Cf., Prop. 2.1 in [16]). *Suppose $\phi: A \rightarrow B$ is a faithfully flat map of Noetherian rings. Then for each prime ideal $P \in \mathcal{P}(A)$ and all integer pairs $(N, r) \in (\mathbb{Z}_{\geq 0})^2$, we have*

$$P^{(N)}B = (PB)^{(N)}, \tag{2.0.1}$$

and $P^{(N)} \subseteq P^r$ if and only if $(PB)^{(N)} = P^{(N)}B \subseteq P^r B = (PB)^r$.

Working over a field \mathbb{F} , we use Proposition (2.6) when $B = A \otimes_{\mathbb{F}} C$ for two affine \mathbb{F} -algebras, so B is an affine \mathbb{F} -algebra; when A and C are domains and \mathbb{F} is algebraically closed, B is a domain, $\mathcal{P}(A) = \text{Spec}(A)$ and $\mathcal{P}(C) = \text{Spec}(C)$.

Theorem 2.7. *For all $n \geq 1$, the symbolic power $(PT + QT)^{(n)} = \sum_{a+b=n} (PT)^{(a)}(QT)^{(b)}$.*

Proof. We'll drop the T 's from the notation, and we will assume that both P, Q are nonzero to justify the effort. For $0 \leq c \leq n$, set $J_c = \sum_{t=0}^c P^{(c-t)}Q^{(t)}$, so $J_c \subseteq J_{c-1}$ for all $1 \leq c \leq n$, since $P^{(c-t)} \subseteq P^{(c-1-t)}$ for $t \leq c-1$ and for $t = c$, $Q^{(c)} \subseteq Q^{(c-1)}$. Note that

$$(P + Q)^n = \sum_{a+b=n} P^a Q^b \subseteq J_n = \sum_{a+b=n} P^{(a)} Q^{(b)} \stackrel{(!)}{\subseteq} (P + Q)^{(n)},$$

and (!) is easy to verify term-by-term for each $P^{(a)}Q^{(b)}$. Indeed, $P^{(a)}Q^{(b)}$ is generated by elements of the form fg with $f \in P^{(a)} \subset R$ and $g \in Q^{(b)} \subset S$ (viewing them as elements of T). We need $fg \in (P+Q)^{(a+b)}$. Per Proposition (2.6), there exist $u \in R - P$ and $v \in S - Q$ such that $uf \in P^a$ and $vg \in Q^b$. Viewing u and v as elements of the overring T , we have $uv \notin (P+Q)$. Indeed, since $P+Q$ is prime, if $uv \in P+Q$, then either u or v is in $P+Q$, but $(P+Q)T \cap R = P$ and $(P+Q)T \cap S = Q$,

contradicting that $u \notin P$ and $v \notin Q$. Therefore, in T , $(uf)(vg) = (uv)(fg) \in P^a Q^b \subset (P+Q)^{a+b}$, which means $fg \in (P+Q)^{(a+b)}$. Thus (!) holds and J_n is a proper ideal.

Since J_n contains $(P+Q)^n$, and $(P+Q)^{(n)}$ is the smallest $(P+Q)$ -primary ideal containing $(P+Q)^n$, the opposite inclusion to (!) will follow once we show that J_n is $(P+Q)$ -primary, i.e., that the set of associated primes $\text{Ass}_T(T/J_n) = \{P+Q\}$. We have short exact sequences of T -modules

$$0 \rightarrow J_{c-1}/J_c \rightarrow T/J_c \rightarrow T/J_{c-1} \rightarrow 0, \quad \text{for all } 1 \leq c \leq n.$$

Thus $\text{Ass}_T(J_{c-1}/J_c) \subseteq \text{Ass}_T(T/J_c) \subseteq \text{Ass}_T(J_{c-1}/J_c) \cup \text{Ass}_T(T/J_{c-1})$ for all $1 \leq c \leq n$, using the fact that given an inclusion of modules $N \subseteq M$,

$$\text{Ass}(N) \subseteq \text{Ass}(M) \subseteq \text{Ass}(N) \cup \text{Ass}(M/N).$$

Thus by iterative unwinding and using that $J_0 = T$, i.e., $\text{Ass}_T(T/J_0) = \emptyset$, we conclude that

$$\emptyset \neq \text{Ass}_T(T/J_n) \subseteq \bigcup_{c=1}^n \text{Ass}_T(J_{c-1}/J_c). \quad (2.0.2)$$

Taking all direct sums and tensor products over \mathbb{F} , we have a series of vector space isomorphisms

$$J_{c-1}/J_c \cong \bigoplus_{a+b=c-1} [P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}], \quad 1 \leq c \leq n. \quad (2.0.3)$$

We prove this first, considering two chains of symbolic powers, where each ideal is expressed as a direct sum of \mathbb{F} -vector spaces:

$$\begin{aligned} P^{(c)} &= V_0 \subseteq P^{(c-1)} = V_0 \oplus V_1 \subseteq \dots \subseteq P^{(0)} = R = V_0 \oplus \dots \oplus V_c, \\ Q^{(c)} &= W_0 \subseteq Q^{(c-1)} = W_0 \oplus W_1 \subseteq \dots \subseteq Q^{(0)} = S = W_0 \oplus \dots \oplus W_c. \end{aligned}$$

In particular, for all pairs $0 \leq a, b \leq c-1$,

$$P^{(a)} = \bigoplus_{i=0}^{c-a} V_i, \quad P^{(a+1)} = \bigoplus_{i=0}^{c-a-1} V_i, \quad Q^{(b)} = \bigoplus_{j=0}^{c-b} W_j, \quad Q^{(b+1)} = \bigoplus_{j=0}^{c-b-1} W_j.$$

For any pair a, b as above with $a+b=c-1$, $c-b=a+1$, and so

$$\bigoplus_{a+b=c-1} \frac{P^{(a)}}{P^{(a+1)}} \otimes_{\mathbb{F}} \frac{Q^{(b)}}{Q^{(b+1)}} \cong \bigoplus_{a+b=c-1} V_{c-a} \otimes W_{c-b} = \bigoplus_{a=0}^{c-1} V_{c-a} \otimes W_{a+1}.$$

We now prove (2.0.3) by killing off a common vector space. First,

$$\begin{aligned} J_{c-1} &= \sum_{a+b=c-1} P^{(a)} Q^{(b)} = \bigoplus_{\substack{0 \leq a \leq c-1 \\ 0 \leq i \leq c-a, 0 \leq j \leq a+1}} V_i \otimes W_j \\ &= \boxed{\bigoplus_{\substack{0 \leq a \leq c-1 \\ 0 \leq i \leq c-a \text{ OR } 0 \leq j \leq a+1}} (V_i \otimes W_j)} \oplus \bigoplus_{a=0}^{c-1} V_{c-a} \otimes W_{a+1}, \\ \text{while } J_c &= \sum_{a+b=c} P^{(a)} Q^{(b)} = \boxed{\bigoplus_{\substack{0 \leq a \leq c \\ 0 \leq i \leq c-a, 0 \leq j \leq a}} V_i \otimes W_j}. \end{aligned}$$

Identifying repeated copies of a $V_i \otimes W_j$ term with $i+j \leq c$ (we can do this since we are working with vector subspaces of the ring T), it is straightforward to check that the boxed sums are equal.

Thus for each $1 \leq c \leq n$, we have canonical isomorphisms of \mathbb{F} -vector spaces:

$$J_{c-1}/J_c \cong \bigoplus_{a=0}^{c-1} V_{c-a} \otimes W_{a+1} \cong \bigoplus_{a+b=c-1} \frac{P^{(a)}}{P^{(a+1)}} \otimes_{\mathbb{F}} \frac{Q^{(b)}}{Q^{(b+1)}}.$$

Therefore, since for each $1 \leq c \leq n$ there is a natural surjective T -module map (hence \mathbb{F} -linear)

$$\bigoplus_{a+b=c-1} [P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}] \rightarrow J_{c-1}/J_c,$$

this map must be injective per isomorphism (2.0.3). Thus for all $1 \leq c \leq n$,

$$\text{Ass}_T(J_{c-1}/J_c) = \bigcup_{a+b=c-1} \text{Ass}_T[P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}].$$

For any $1 \leq c \leq n$ such that $J_{c-1}/J_c \neq 0$, i.e., $\text{Ass}_T(J_{c-1}/J_c) \neq \emptyset$, in turn the above identity implies that one of the modules $P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}$ is nonzero, in which case

$$\text{Ass}_T(J_{c-1}/J_c) = \bigcup_{a+b=c-1} \text{Ass}_T[P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}] = \{P + Q\}. \quad (2.0.4)$$

To explain the right-hand equality: for any pair $(a, b) \in (\mathbb{Z}_{\geq 0})^2$, Proposition (2.5) says that

$$M_{a,b} := P^{(a)}/P^{(a+1)} \otimes Q^{(b)}/Q^{(b+1)}$$

is a finitely generated torsion-free module over $T' = (R/P) \otimes (S/Q) \cong T/(P + Q)$; thus when $M_{a,b} \neq 0$, we have $\text{Ass}_{T/(P+Q)}(M_{a,b}) = \{(0)\}$ by Lemma (2.2), that is, $\text{Ass}_T(M_{a,b}) = \{P + Q\}$.

Finally, per (2.0.4) and the inclusion (2.0.2) for $\text{Ass}_T(T/J_n) \neq \emptyset$ —note inclusion (!) implies that J_n is proper, we have $\text{Ass}_T(T/J_n) = \bigcup_{c=1}^n \text{Ass}_T(J_{c-1}/J_c) = \{P + Q\}$, that is, the ideal J_n is $(P + Q)$ -primary. Thus $J_n \supseteq (P + Q)^{(n)}$, and indeed this is an equality. \square

We now deduce a multinomial theorem by induction on the number of tensor factors:

Theorem 2.8. *Let \mathbb{F} be an algebraically closed field. Let R_1, \dots, R_n ($n \geq 2$) be affine commutative \mathbb{F} -algebras which are domains. Fix any n prime ideals P_i in R_i , and consider the expanded ideals $P'_i = P_i T$ in the affine domain $T = (\bigotimes_{\mathbb{F}})_{i=1}^n R_i$. Then the symbolic power*

$$\left(\sum_{i=1}^n P'_i \right)^{(N)} = \sum_{A_1 + \dots + A_n = N} \prod_{i=1}^n (P'_i)^{(A_i)} \text{ for any } N \geq 0. \quad (2.0.5)$$

Proof. Induce on the number n of tensor factors with base case $n = 2$ being Theorem (2.7). Now suppose $n \geq 3$, and assume the result for tensoring up to $n - 1$ factors. Suppose that $R = R_1$ and $S = R_2 \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} R_n$, and that we have an expansion result in S of the form

$$\left(\sum_{i=2}^n P_i \right)^{(N)} = \sum_{A_2 + \dots + A_n = N} \prod_{i=2}^n P_i^{(A_i)} \text{ for all nonnegative integers } N \quad (2.0.6)$$

for $n - 1$ primes $P_i \subseteq R_i$ ($2 \leq i \leq n$). The sum $Q := \sum_{i=2}^n P_i$ is prime along with all extensions of the P_i to S . Given a prime $P = P_1$ in R , the sum $P + Q$ is prime in $T = R \otimes_{\mathbb{F}} S$, together with

all extensions $P_i T$ and $Q T$ being prime. The first equality below holds by Theorem (2.7), and per Proposition (2.6), applied to the extension $\phi: S \hookrightarrow T$, the second equality holds by (2.0.6):

$$\begin{aligned} (P + Q)^{(N)} &= \sum_{A_1+B=N} P^{(A_1)} Q^{(B)} = \sum_{A_1=0}^N P_1^{(A_1)} \left(\sum_{A_2+\dots+A_n=N-A_1} \prod_{i=2}^n P_i^{(A_i)} \right) \\ &\subseteq \sum_{A_1+A_2+\dots+A_n=N} \prod_{i=1}^n P_i^{(A_i)}, \end{aligned}$$

using the fact that $I(J + K) \subseteq IJ + IK$ whenever I, J, K are ideals in a commutative ring. This proves the n -fold version of the hard inclusion in the proof of Theorem (2.7); deducing the opposite inclusion is about as easy as before, hence the above inclusion is an equality. \square

Proving Theorem (1.2): We use the Multinomial Theorem (2.8) to deduce a corollary; Theorem (1.2) is the uniform growth bound analogue.

Corollary 2.9. *Let \mathbb{F} be an algebraically closed field. Let R_1, \dots, R_n ($n \geq 2$) be affine commutative \mathbb{F} -algebras which are domains. Fix n primes $P_i \subseteq R_i$, and consider the expanded ideals $P'_i = P_i T$ in the affine domain $T = (\bigotimes_{\mathbb{F}})_{i=1}^n R_i$; set $Q = \sum_{i=1}^n P'_i$. Suppose that for each $1 \leq i \leq n$, there exists a positive integer D_i such that either:*

- (1) $P_i^{(D_i r)} \subseteq P_i^r$ for all $r > 0$; or, even stronger,
- (2) $P_i^{(D_i(r-1)+1)} \subseteq P_i^r$ for all $r > 0$.

When (1) holds for all i , $Q^{(Dr)} \subseteq Q^r$ for all $r > 0$, where $D = D_1 + \dots + D_n$. When (2) holds for all i , $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$, where $D = \max\{D_1, \dots, D_n\}$.

Proof. Assume (1) holds for all i . Per Theorem (2.8) note that for $D = D_1 + D_2 + \dots + D_n$,

$$Q^{(Dr)} = \sum_{A_1+A_2+\dots+A_n=D_1 r + D_2 r + \dots + D_n r} \prod_{i=1}^n (P'_i)^{(A_i)}.$$

In each n -tuple of indices (A_1, \dots, A_n) , we must have that $A_j \geq D_j r$ for some j , otherwise $\sum_{i=1}^n A_i < \sum_{i=1}^n D_i r$, a contradiction. Thus each summand $\prod_{i=1}^n (P'_i)^{(A_i)}$ will lie in some $(P'_j)^r$ applying (1) and Proposition (2.6), and hence also in Q^r . Since $r > 0$ was arbitrary, we win.

If (2) holds for all i , then $P_i^{(D(r-1)+1)} \subseteq P_i^r$ for all $r > 0$ and all i , where $D = \max_{1 \leq i \leq n} D_i$, so equivalently per Lemma (2.1) and Proposition (2.6), for all n -tuples $(A_1, \dots, A_n) \in (\mathbb{Z}_{\geq 0})^n$, we have $(P'_i)^{(A_i)} \subseteq (P'_i)^{\lceil A_i/D \rceil} \subseteq Q^{\lceil A_i/D \rceil}$. For all nonnegative integers N , per Theorem (2.8)

$$Q^{(N)} = \sum_{A_1+\dots+A_n=N} \prod_{i=1}^n (P'_i)^{(A_i)} \subseteq \sum_{A_1+\dots+A_n=N} \prod_{i=1}^n (P'_i)^{\lceil A_i/D \rceil} \subseteq \sum_{A_1+\dots+A_n=N} \prod_{i=1}^n Q^{\lceil A_i/D \rceil} \subseteq Q^{\lceil N/D \rceil},$$

since the integer $\sum_{i=1}^n \lceil A_i/D \rceil \geq \lceil (\sum_{i=1}^n A_i)/D \rceil = \lceil N/D \rceil$ for all n -tuples $(A_1, \dots, A_n) \in (\mathbb{Z}_{\geq 0})^n$ with $\sum_{i=1}^n A_i = N$. Thus equivalently, $Q^{(D(r-1)+1)} \subseteq Q^r$ for all $r > 0$ by Lemma (2.1). \square

Remark 2.10. Let $R = \mathbb{F}[x_1, \dots, x_m]$, $S = \mathbb{F}[y_1, \dots, y_n]$, and $T = R \otimes_{\mathbb{F}} S \cong \mathbb{F}[x_1, \dots, x_m, y_1, \dots, y_n]$ be polynomial rings over a field \mathbb{F} . Our original inspiration for Theorem (2.8) was the following

Theorem (Thm 7.8 of Bocci et al [4]). Let $I \subseteq R$ and $J \subseteq S$ be squarefree monomial ideals. Let $I' = IT$ and $J' = JT$ be their expansions to T . Then for any $N \geq 0$, the symbolic power

$$(I' + J')^{(N)} = \sum_{i=0}^N (I')^{(N-i)} (J')^{(i)} = \sum_{A+B=N} (I')^{(A)} (J')^{(B)}. \quad (2.0.7)$$

Ha, Nguyen, Trung, and Trung [7, Thm 3.4] recently extended the above theorem to the case of two nonzero ideals $I \subseteq R, J \subseteq S$ in two Noetherian commutative \mathbb{F} -algebras such that $T = R \otimes_{\mathbb{F}} S$ is also Noetherian. A general multinomial theorem then follows by adapting the proof of Theorem (2.8), where one containment would require the n -fold version of [7, Lem 2.1(i)]. Leveraging this multinomial expansion, together with the general versions of Lemma (2.1) and Proposition (2.6) proved in [16, Prop 2.1, Lem 3.3, Prop 3.4], one can extend Corollary (2.9) to allow any proper ideals $I_i \subseteq R_i$, though we must still assume the R_i are affine domains over an algebraically closed field \mathbb{F} , in deference to the $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ example from earlier. **Note(!)**: the proof of [16, Prop 2.1] will work up to a tweak of multiplicative system, for those who define symbolic powers of proper ideals using only minimal associated primes as in [7], rather than using all associated primes as in [16].

Remark 2.11. If R is a Noetherian ring of dimension at least 2, then $\text{Spec}(R)$ is infinite; indeed, any height two prime ideal can be expressed as a union of height one primes, but not of finitely many [2, Exer 21.11]. To see this bound is sharp, note that $\#\text{Spec}(R) = 2$ when R is any discrete valuation ring (i.e., regular local ring of dimension 1); see also [2, Exer 21.12]. Thus if R_1, \dots, R_n are a collection of \mathbb{F} -affine domains of dimension at least two, with \mathbb{F} algebraically closed, and the domain $T = (\bigotimes_{\mathbb{F}})_{i=1}^n R_i$, then for every $1 \leq h \leq n$, there are infinitely-many primes in T of height h ; indeed $\mathcal{Q}_{ED}(T) := \{Q = \sum_{i=1}^n P_i T \in \text{Spec}(T) : \forall 1 \leq i \leq n, P_i \in \text{Spec}(R_i)\}$ is an infinite set.

3. FINALE: SAMPLE APPLICATIONS TO TENSOR POWER DOMAINS

We begin with two results on uniform linear bounds on asymptotic growth of symbolic powers for equicharacteristic Noetherian domains that have nice structure, but need not be regular. The first is due to Huneke-Katz-Validashti, while the second is due to Ajinkya A. More.

Theorem 3.1 (Cor 3.10 in [11]). *Let R be an equicharacteristic local domain such that R is an isolated singularity. Assume that R is either essentially of finite type over a field of characteristic zero or R has positive characteristic, is F -finite and analytically irreducible. Then there exists an $E \geq 1$ such that $P^{(Er)} \subseteq P^r$ for all $r > 0$ and all prime ideals P .*

Theorem 3.2 ([14], Thm 4.4 and Cor 4.5). *Suppose $R \subseteq S$ is a finite extension of equicharacteristic normal domains such that: (1) S is a regular ring generated as an R -module by n elements, and $n!$ is invertible in S ; and (2) R is either essentially of finite type over an excellent Noetherian local ring (or over \mathbb{Z}), or is characteristic $p > 0$ and F -finite. Then there exists an $E \geq 1$ such that $P^{(Er)} \subseteq P^r$ for all $r > 0$ and all prime ideals P in R .*

Some remarks over \mathbb{C} : First, we note that if R is the coordinate ring of a complex affine variety whose singular locus is zero dimensional, then in tandem with the results of Ein-Lazarsfeld-Smith and Hochster-Huneke [6, 9], Theorem (3.1) would yield a uniform slope E for all primes in R . In particular, $\mathbb{C}[x, y, z]/(y^2 - xz)$ and $\mathbb{C}[x, y, z, w]/(xy - zw)$ are covered by Theorem (3.1). Meanwhile, if $S = \mathbb{C}[x_1, \dots, x_n]$ is a polynomial ring, then Theorem (3.2) applies when R is any Veronese subring of S (i.e., the subalgebra generated by all monomials of a fixed degree in the x_i).

Fix an algebraically closed field \mathbb{F} . If A is an \mathbb{F} -affine domain, i.e., finitely generated as an \mathbb{F} -algebra, we use the tensor power notation $R = A^{\otimes N} = (\bigotimes_{\mathbb{F}})_{i=1}^N A_i$ to denote the \mathbb{F} -affine ring obtained by tensoring together N copies of A over \mathbb{F} , where A_i and A_j are presented as quotients of polynomial rings in disjoint sets of variables when $i < j$. We define a set of prime ideals $\mathcal{Q}_{ED}(R) = \{Q = \sum_{i=1}^N P_i R \in \text{Spec}(R) : \text{each } P_i \in \text{Spec}(A_i)\}$ to which Corollary (2.9) applies.

Example 3.3. *The rings $\mathbb{C}[x, y, z]/(y^2 - xz)$ and $\mathbb{C}[x, y, z, w]/(xy - zw)$ above fall under a more general setup: to start, given integers a and d both at least two, consider an affine hypersurface domain $A = \mathbb{C}[z_1, \dots, z_a]/(F_d(\bar{z}))$ where F_d is a homogeneous polynomial of degree d , with isolated singularity at the origin. Let $V_A = \text{Spec}(A) \subseteq \mathbb{C}^a$ and $V = \text{Spec}(R) \subseteq \mathbb{C}^{aN}$ where*

$$R = A^{\otimes N} = \frac{\mathbb{C}[z_{i,1}, \dots, z_{i,a} : 1 \leq i \leq N]}{(F_d(z_{i,1}, \dots, z_{i,a}) : 1 \leq i \leq N)}.$$

Per Theorem (3.1), there is a positive integer E such that $P^{(Er)} \subseteq P^r$ for all $r > 0$ and all primes P in A , so Corollary (2.9) says that $Q^{(NE \cdot r)} \subseteq Q^r$ for all $r > 0$ and all primes $Q \in \mathcal{Q}_{ED}(R)$. Meanwhile, in terms of n -factor Cartesian products, the singular locus

$$\text{Sing}(V) = (\{0\} \times V_A \times \dots \times V_A) \cup (V_A \times \{0\} \times V_A \times \dots \times V_A) \cup \dots \cup (V_A \times \dots \times V_A \times \{0\})$$

is equidimensional of dimension $(a-1)(N-1)$. In particular, although R is not an isolated singularity for $N \geq 2$, there are still uniform linear bounds lurking for the asymptotic growth of symbolic powers of primes in R .

As a variant of this example, given an integer $D \geq 2$, one starts with $A = V_D$ being the D -th Veronese subring of an affine polynomial ring over \mathbb{C} . Theorem (3.2) implies that a uniform E exists for prime ideals in A , while ([16], Thm 4.3)—for the monomial prime ideals of A —yields a tight lower bound on E , namely that $E \geq D$ when $\dim(A) \geq 2$. This tight bound would carry over to all the rings $R = A^{\otimes N}$ as well per Proposition (2.6). As a separate variant of Example (3.3), in [16] we show that if a normal toric ring A corresponds to a **simplicial** affine toric variety over an algebraically closed field \mathbb{F} , i.e., its divisor class group is finite, then the group order D works for case (2) in Corollary (2.9), for all primes $Q = \sum_{i=1}^N P'_i \in \mathcal{Q}_{ED}(A^{\otimes N})$, where each prime $P = P_i$ is either *normally torsion free* (i.e., $P^{(r)} = P^r$ for all $r > 0$), or of *height one* with $P^{(r)} \neq P^r$ for some $r > 1$. This covers all $Q \in \mathcal{Q}_{ED}(A^{\otimes N})$ when A is any two-dimensional normal toric ring.

Closing Remarks. Launching from Theorem (1.1) in the introduction, we have deduced a more powerful criterion for proliferating uniform linear bounds on the growth of symbolic powers of prime ideals (e.g., Harbourne-Huneke bounds)—Theorem (1.2). This criterion contributes further evidence for Huneke’s philosophy in [10] about uniform bounds lurking throughout commutative algebra; it covers a reasonably **prodigious** class of examples involving domains of finite type over a field we assume is algebraically closed, both to assuage valid concerns and to streamline arguments. We close with a goalpost question that exceeds our grasp at present: Given the role of tensor products in our manuscript, do analogues of the above criteria hold for other product constructions in commutative algebra, such as Segre products of \mathbb{N} -graded rings, or fiber products of toric rings?

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